Binomial [77 marks]

1. [Maximum mark: 4]

Find the coefficient of x^8 in the expansion of $(2x - 5)^{11}$. [4]

Markscheme

EITHER

attempt to form a product of binomial coefficient, a power of $2x$ and a power of -5 seen *(M1)*

$^{11}C_8(2x)^8(-5)^3$ OR $^{11}C_3(2x)^8(-5)^3$ OR $165 \times (2x)^8(-5)^3$. *(A1)(A1)*

Note: Award *A1* for $^{11}C_8$ or $^{11}C_3$ or 165, *A1* for $(2x)^8(-5)^3$.

OR

attempt to use the general term *(M1)*

$^{11}C_r(2x)^{11-r}(-5)^r$ and $r = 3$ *(A1)(A1)*

THEN

-5280000 (exact) *A1*

Note: Award *A0* for a final answer of $-5280000x^8$.

[4 marks]

2. [Maximum mark: 4]

Find the coefficient of x^6 in the expansion of $(2x - 5)^9$. [4]

Markscheme

EITHER

attempt to form a product of binomial coefficient, a power of $2x$ and a power of -5 seen ***(M1)***

$^9C_3(2x)^6(-5)^3$ OR $^9C_6(2x)^6(-5)^3$ OR $84 \times (2x)^6(-5)^3$ ***(A1)(A1)***

Note: Award ***A1*** for $^9C_6$ or $^9C_3$ or 84, ***A1*** for $(2x)^6(-5)^3$.

OR

attempt to use the general term ***(M1)***

$^9C_r(2x)^{9-r}(-5)^r$ and $r = 3$ ***(A1)(A1)***

THEN

-672000 (exact) ***A1***

Note: Award ***A0*** for a final answer of $-672000x^6$.

[4 marks]

3. [Maximum mark: 6]

The binomial expansion of $(1 + kx)^n$ is given by

$1 + \frac{9x}{2} + 15k^2x^2 + \ldots + k^nx^n$, where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of *n* and the value of *k*. [6]

Markscheme

attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1kx + {}^nC_2k^2x^2 + \ldots$ OR ${}^nC_1k = \frac{9}{2}$ OR ${}^nC_2 = 15$

$nk = \frac{9}{2}$ ***(A1)***

$n\frac{(n-1)}{2} = 15$ OR $\frac{n!}{(n-2)!2!} = 15$ ***(A1)***

$\left(n^2 - n - 30 = 0\right)$ OR $n(n-1) = 30$

valid attempt to solve ***(M1)***

$(n-6)(n-5) = 0$ OR $6(6-1) = 30$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 6$ ***A1***

$\Rightarrow k = \frac{3}{4}$ ***A1***

Note: If candidate finds $n = 6$ with no working shown, award ***M1A0A0M1A1A0***.

If candidate finds $n = 6$ and $k = \frac{3}{4}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

4. [Maximum mark: 6]

The binomial expansion of $(1+kx)^n$ is given by $1 + 12x + 28k^2x^2 + \ldots + k^nx^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of n and the value of k. [6]

Markscheme

attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1kx + {}^nC_2k^2x^2 + \ldots$ OR ${}^nC_1k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve ***(M1)***

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \frac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0***.

If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

5. [Maximum mark: 7]

The coefficient of x^6 in the expansion of $\left(ax^3 + b\right)^8$ is 448.

The coefficient of x^6 in the expansion of $\left(ax^3 + b\right)^{10}$ is 2880.

Find the value of a and the value of b, where a, $b > 0$. [7]

Markscheme

product of a binomial coefficient, a power of ax^3 and a power of b seen ***(M1)***

evidence of correct term chosen

for $n = 8 : r = 2$ (or $r = 6$) OR for $n = 10 : r = 2$ (or $r = 8$) ***(A1)***

correct equations (may include powers of x) ***A1A1***

${}_8C_2 a^2 b^6 = 448 \left(28a^2b^6 = 448 \Rightarrow a^2b^6 = 16\right)$,
${}_{10}C_2 a^2 b^8 = 2880 \left(45a^2b^8 = 2880 \Rightarrow a^2b^8 = 64\right)$

attempt to solve their system in a and b algebraically or graphically ***(M1)***

$b = 2 ; a = \frac{1}{2}$ ***A1A1***

Note: Award a maximum of ***(M1)(A1)A1A1(M1)A1A0*** for $b = \pm 2$ and/or $a = \pm\frac{1}{2}$.

[7 marks]

6. [Maximum mark: 6]

Consider the expansion of $\frac{(ax+1)^9}{21x^2}$, where $a \neq 0$. The coefficient of the term in x^4 is $\frac{8}{7}a^5$.

Find the value of a. [6]

Markscheme

Note: Do not award any marks if there is clear evidence of adding instead of multiplying, for example ${}^9C_r + (ax)^{9-r} + (1)^r$.

valid approach for expansion (must be the product of a binomial coefficient with $n = 9$ and a power of ax) ***(M1)***

${}^9C_r(ax)^{9-r}(1)^r$ OR ${}^9C_{9-r}(ax)^r(1)^{9-r}$ OR ${}^9C_0(ax)^0(1)^9 + {}^9C_1(ax)^1(1)^8 + \ldots$

recognizing that the term in x^6 is needed ***(M1)***

$\frac{\text{Term in } x^6}{21x^2} = kx^4$ OR $r = 6$ OR $r = 3$ OR $9 - r = 6$

correct term or coefficient in binomial expansion (seen anywhere) ***(A1)***

${}^9C_6(ax)^6(1)^3$ OR ${}^9C_3a^6x^6$ OR $84\left(a^6x^6\right)(1)$ OR $84a^6$

EITHER

correct term in x^4 or coefficient (may be seen in equation) ***(A1)***

$\frac{^9C_6}{21}a^6x^4$ OR $4a^6x^4$ OR $4a^6$

Set their term in x^4 or coefficient of x^4 equal to $\frac{8}{7}a^5x^4$ or $\frac{8}{7}a^5$ (do not accept other powers of x) ***(M1)***

$\frac{^9C_3}{21}a^6x^4 = \frac{8}{7}a^5x^4$ OR $4a^6 = \frac{8}{7}a^5$

OR

correct term in x^6 or coefficient of x^6 (may be seen in equation) ***(A1)***

$84a^6x^6$ OR $84a^6$

set their term in x^6 or coefficient of x^6 equal to $24a^5x^6$ or $24a^5$ (do not accept other powers of x) ***(M1)***

$84a^6x^6 = 24a^5x^6$ OR $84a = 24$

THEN

$a = \frac{2}{7} \approx 0.286\,(0.285714\ldots)$ ***A1***

Note: Award **A0** for the final mark for $a = \frac{2}{7}$ and $a = 0$.

[6 marks]

7. [Maximum mark: 15]

(a.i) Expand and simplify $(1 - a)^3$ in ascending powers of a. [2]

Markscheme

EITHER

attempt to use binomial expansion *(M1)*

$1 + {}_3C_1 \times 1 \times (-a) + {}_3C_2 \times 1 \times (-a)^2 + 1 \times (-a)^3$

OR

$(1-a)(1-a)(1-a)$

$= (1-a)(1-2a+a^2)$ *(M1)*

THEN

$= 1 - 3a + 3a^2 - a^3$ *A1*

[2 marks]

(a.ii) By using a suitable substitution for a, show that
$1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x = 8\sin^6 x.$ [4]

Markscheme

$a = \cos 2x$ *(A1)*

So, $1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x =$

$(1 - \cos 2x)^3$ *A1*

attempt to substitute any double angle rule for $\cos 2x$ into $(1 - \cos 2x)^3$
(M1)

$= (2\sin^2 x)^3$ *A1*

$= 8\sin^6 x$ *AG*

Note: Allow working RHS to LHS.

[4 marks]

Consider $f(x) = 4\cos x\left(1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\right)$.

(b.i) Show that $\int_0^m f(x)\,\mathrm{d}\,x = \frac{32}{7}\sin^7 m$, where m is a positive real constant. [4]

Markscheme

recognizing to integrate $\int\left(4\cos x \times 8\sin^6 x\right)\mathrm{d}\,x$ ***(M1)***

EITHER

applies integration by inspection ***(M1)***

$32\int\left(\cos x \times (\sin x)^6\right)\mathrm{d}\,x$

$= \frac{32}{7}\sin^7 x\,(+c)$ ***A1***

$\left[\frac{32}{7}\sin^7 x\right]_0^m \quad \left(= \frac{32}{7}\sin^7 m - \frac{32}{7}\sin^7 0\right)$ ***A1***

OR

$u = \sin x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$ ***(M1)***

$\int 32\cos x\left(\sin^6 x\right)\mathrm{d}\,x = \int 32u^6\,\mathrm{d}\,u$

$= \frac{32}{7}u^7\,(+c)$ ***A1***

$\left[\frac{32}{7}\sin^7 x\right]_0^m$ OR $\left[\frac{32}{7}u^7\right]_0^{\sin m} \quad \left(= \frac{32}{7}\sin^7 m - \frac{32}{7}\sin^7 0\right)$ ***A1***

THEN

$= \frac{32}{7}\sin^7 m$ ***AG***

[4 marks]

(b.ii) It is given that $\int_m^{\frac{\pi}{2}} f(x)\,\mathrm{d}\,x = \frac{127}{28}$, where $0 \leq m \leq \frac{\pi}{2}$. Find the value of m. [5]

Markscheme

EITHER

$\int_m^{\frac{\pi}{2}} f(x)\,\mathrm{d}\,x \left(= \left[\frac{32}{7}\sin^7 x\right]_m^{\frac{\pi}{2}}\right) = \frac{32}{7}\sin^7 \frac{\pi}{2} - \frac{32}{7}\sin^7 m$ ***M1***

$\frac{32}{7}\sin^7 \frac{\pi}{2} - \frac{32}{7}\sin^7 m = \frac{127}{28}$ OR $\frac{32}{7}\left(1 - \sin^7 m\right) = \frac{127}{28}$ ***(M1)***

OR

$\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}\,x = \int_0^m f(x)\,\mathrm{d}\,x + \int_m^{\frac{\pi}{2}} f(x)\,\mathrm{d}\,x$ ***M1***

$\frac{32}{7} = \frac{32}{7}\sin^7 m + \frac{127}{28}$ ***(M1)***

THEN

$\sin^7 m = \frac{1}{128} \quad \left(= \frac{1}{2^7}\right)$ ***(A1)***

$\sin m = \frac{1}{2}$ ***(A1)***

$m = \frac{\pi}{6}$ ***A1***

[5 marks]

8. [Maximum mark: 7]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a,\ b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. [2]

Markscheme

EITHER

recognises the required term (or coefficient) in the expansion ***(M1)***

$bx^5 = {}_7C_2x^5 1^2$ OR $b = {}_7C_2$ OR ${}_7C_5$

$b = \frac{7!}{2!5!} \left(= \frac{7!}{2!(7-2)!}\right)$

correct working ***A1***

$\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times5\times4\times3\times2\times1}$ OR $\frac{7\times6}{2!}$ OR $\frac{42}{2}$

OR

lists terms from row 7 of Pascal's triangle ***(M1)***

$1,\ 7,\ 21, \ldots$ ***A1***

THEN

$b = 21$ ***AG***

[2 marks]

(b) The third term in the expansion is the mean of the second term and the fourth term in the expansion.

Find the possible values of x. [5]

Markscheme

$a = 7$ ***(A1)***

correct equation ***A1***

$21x^5 = \frac{ax^6+35x^4}{2}$ OR $21x^5 = \frac{7x^6+35x^4}{2}$

correct quadratic equation ***A1***

$7x^2 - 42x + 35 = 0$ OR $x^2 - 6x + 5 = 0$ (or equivalent)

valid attempt to solve **their** quadratic ***(M1)***

$(x-1)(x-5) = 0$ OR $x = \frac{6 \pm \sqrt{(-6)^2-4(1)(5)}}{2(1)}$

$x = 1,\ x = 5$ ***A1***

Note: Award final ***A0*** for obtaining $x = 0,\ x = 1,\ x = 5$.

[5 marks]

9. [Maximum mark: 5]

In the expansion of $(x+k)^7$, where $k \in \mathbb{R}$, the coefficient of the term in x^5 is 63.

Find the possible values of k. [5]

Markscheme

EITHER

attempt to use the binomial expansion of $(x+k)^7$ ***(M1)***

${}^7C_0x^7k^0 + {}^7C_1x^6k^1 + {}^7C_2x^5k^2 + \ldots$ (or ${}^7C_0k^7x^0 + {}^7C_1k^5x^1 + {}^7C_2k^5x^2 + \ldots$)

identifying the correct term ${}^7C_2x^5k^2$ (or ${}^7C_5k^2x^5$) ***(A1)***

OR

attempt to use the general term ${}^7C_r x^r k^{7-r}$ (or ${}^7C_r k^r x^{7-r}$) ***(M1)***

$r = 2$ (or $r = 5$) ***(A1)***

THEN

${}^7C_2 = 21$ (or ${}^7C_5 = 21$ (seen anywhere) ***(A1)***

$21x^5k^2 = 63x^5 \ \left(21k^2 = 63\ ,\ k^2 = 3\right)$ ***A1***

$k = \pm\sqrt{3}$ ***A1***

Note: If working shown, award ***M1A1A1A1A0*** for $k = \sqrt{3}$.

[5 marks]

10. [Maximum mark: 5]

Consider the expansion of $\left(3 + x^2\right)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of x^4 is $20\,412$, find the value of n. [5]

Markscheme

METHOD 1

product of a binomial coefficient, a power of 3 (and a power of x^2) seen ***(M1)***

evidence of correct term chosen ***(A1)***

${}^{n+1}C_2 \times 3^{n+1-2} \times \left(x^2\right)^2 \left(= \frac{n(n+1)}{2} \times 3^{n-1} \times x^4\right)$ **OR** $n - r = 1$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

EITHER

$^{n+1}C_2 \times 3^{n-1} = 20412$ *(A1)*

OR

$^{r+2}C_r \times 3^r = 20412 \Rightarrow r = 6$ *(A1)*

THEN

$n = 7$ *A1*

METHOD 2

$3^{n+1}\left(1 + \frac{x^2}{3}\right)^{n+1}$

product of a binomial coefficient, and a power of $\frac{x^2}{3}$ **OR** $\frac{1}{3}$ seen *(M1)*

evidence of correct term chosen *(A1)*

$3^{n+1} \times \frac{n(n+1)}{2!} \times \left(\frac{x^2}{3}\right)^2 \left(= \frac{3^{n-1}}{2} n(n+1)x^4\right)$

equating their coefficient to 20412 or their term to $20412x^4$ *(M1)*

$3^{n-1} \times \frac{n(n+1)}{2} = 20412$ *(A1)*

$n = 7$ *A1*

[5 marks]

11. [Maximum mark: 6]

Consider the expansion of $\left(3x^2 - \frac{k}{x}\right)^9$, where $k > 0$.

The coefficient of the term in x^6 is 6048. Find the value of k. [6]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid approach for expansion (must have correct substitution for parameters, but accept an incorrect value for r). ***(M1)***

eg $\binom{9}{r}\left(3x^2\right)^{9-r}\left(-\frac{k}{x}\right)^r$, $\left(3x^2\right)^9 + \binom{9}{1}\left(3x^2\right)^8\left(-\frac{k}{x}\right)^1 + \binom{9}{2}\left(3x^2\right)^7\left(-\frac{k}{x}\right)^2 + \ldots$

valid attempt to identify correct term ***(M1)***

eg $2(9-r) - r = 6$, $\left(x^2\right)^r\left(x^{-1}\right)^{9-r} = x^6$

identifying correct term (may be indicated in expansion) ***(A1)***

eg $r = 4,\ r = 5$

correct term or coefficient in binominal expansion ***(A1)***

eg $\binom{9}{4}\left(3x^2\right)^5\left(-\frac{k}{x}\right)^4$, $126\left(243x^{10}\right)\left(\frac{k^4}{x^4}\right),\ 30618k^4$

correct equation in k ***(A1)***

eg $\binom{9}{4}(243)\left(k^4\right)x^6 = 6048x^6$, $30618k^4 = 6048$

$k = \frac{2}{3}$ (exact) 0.667 ***A1 N3***

Note: Do not award ***A1*** if additional answers given.

[6 marks]

12. [Maximum mark: 6]

Consider $\binom{11}{a} = \frac{11!}{a!\,9!}$.

(a) Find the value of a. [2]

Markscheme

valid approach ***(M1)***

eg $11 - a = 9$, $\frac{11!}{9!(11-9)!}$

$a = 2$ ***A1 N2***

[2 marks]

(b) Hence or otherwise find the coefficient of the term in x^9 in the expansion of $(x+3)^{11}$. [4]

Markscheme

valid approach for expansion using $n = 11$ ***(M1)***

eg $\binom{11}{r}x^{11-r}3^r$, $a^{11}b^0 + \binom{11}{1}a^{10}b^1 + \binom{11}{2}a^9b^2 + \ldots$

evidence of choosing correct term ***A1***

eg $\binom{11}{2}3^2$, $\binom{11}{2}x^9 3^2$, $\binom{11}{9}3^2$

correct working for binomial coefficient (seen anywhere, do not accept factorials) ***A1***

eg 55, $\binom{11}{2} = 55$, 55×3^2, $(55 \times 9)x^9$, $\frac{11\times10}{2} \times 9$

495 ***A1 N2***

Note: If there is clear evidence of adding instead of multiplying, award ***A1*** for the correct working for binomial coefficient, but no other marks. For example, $55x^9 \times 3^2$ would earn ***M0A0A1A0***.

Do not award final ***A1*** for a final answer of $495x^9$, even if 495 is seen previously. If no working shown, award ***N1*** for $495x^9$.

[4 marks]